6-th, 8-th and 10-th order Hermite Integration Scheme for N-Body System

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1 6-th order

Taylor expansions for acceleration and its derivatives are expressed as:

$$F_1 = F_0 + F_0^{(1)}\Delta + \frac{F_0^{(2)}}{2!}\Delta^2 + \frac{F_0^{(3)}}{3!}\Delta^3 + \frac{F_0^{(4)}}{4!}\Delta^4 + \frac{F_0^{(5)}}{5!}\Delta^5 \quad (1)$$

$$F_1^{(1)} = F_0^{(1)} + F_0^{(2)}\Delta + \frac{F_0^{(3)}}{2!}\Delta^2 + \frac{F_0^{(4)}}{3!}\Delta^3 + \frac{F_0^{(5)}}{4!}\Delta^4 \quad (2)$$

$$F_1^{(2)} = F_0^{(2)} + F_0^{(3)}\Delta + \frac{F_0^{(4)}}{2!}\Delta^2 + \frac{F_0^{(5)}}{3!}\Delta^3 \quad (3)$$

$$(4)$$

where $F_0^{(n)}$ and $F_1^{(n)}$ are n-th derivatives at t_0 and t_1, respectively and $\Delta = t_1 - t_0$.

We solve these equations for $F_0^{(3)}, F_0^{(4)}, F_0^{(5)}$ to obtain approximation formulae as follows. First, we arrange the equations into matrix form:

$$\begin{pmatrix} \Delta^3/3! & \Delta^4/4! & \Delta^5/5! \\ \Delta^2/2! & \Delta^3/3! & \Delta^4/4! \\ \Delta & \Delta^2/2! & \Delta^3/3! \end{pmatrix} \begin{pmatrix} F_0^{(3)} \\ F_0^{(4)} \\ F_0^{(5)} \end{pmatrix} = \begin{pmatrix} F - (F_0 + F_0^{(1)}\Delta + F_0^{(2)}/2!\Delta^2) \\ F^{(1)} - (F_0^{(1)} + F_0^{(2)}\Delta) \\ F^{(2)} - F_0^{(2)} \end{pmatrix} \quad (5)$$

This equation has the following solutions.

$$\Delta^3 F_0^{(3)} = -3(20(F_0^{(0)} - F_1^{(0)}) + (12F_0^{(1)} + 8F_1^{(1)})\Delta + (3F_0^{(2)} - F_1^{(2)})\Delta^2) \quad (6)$$

$$\Delta^4 F_0^{(4)} = 12(30(F_0^{(0)} - F_1^{(0)}) + (16F_0^{(1)} + 14F_1^{(1)})\Delta + (3F_0^{(2)} - 2F_1^{(2)})\Delta^2) \quad (7)$$

$$\Delta^5 F_0^{(4)} = -60(12(F_0^{(0)} - F_1^{(0)}) + 6(F_0^{(1)} + F_1^{(1)})\Delta + (F_0^{(2)} - F_1^{(2)})\Delta^2) \quad (8)$$

Similarly, we can estimate r and v at t_1 with the Taylor expiation upto $F^{(5)}$.

$$r_1 = r_0 + v_0\Delta + \sum_{i=2}^{7} \frac{F_0^{(i-1)}}{n!}\Delta^n, v_1 = v_0 + \sum_{i=1}^{6} \frac{F_0^{(i-1)}}{n!}\Delta^n \quad (9)$$

Put $F^{(3)}$, $F^{(4)}$, $F^{(5)}$ and we have

$$r_1 = r_0 + v_0\Delta + (\frac{5}{14}F_0^{(0)} + \frac{1}{7}F_1^{(0)})\Delta^2 + (\frac{13}{210}F_0^{(1)} - \frac{4}{105}F_1^{(1)})\Delta^3 + (\frac{1}{210}F_0^{(2)} + \frac{1}{280}F_1^{(2)})\Delta^4 \quad (10)$$

$$v_1 = v_0 + \frac{1}{2}(F_0^{(0)} + F_1^{(0)})\Delta + \frac{1}{10}(F_0^{(1)} - F_1^{(1)})\Delta^2 + \frac{1}{120}(F_0^{(2)} + F_1^{(2)})\Delta^3 \quad (11)$$

Alternatively, we have the implicit form as

$$v_1 = v_0 + \frac{1}{2}(F_0^{(0)} + F_1^{(0)})\Delta + \frac{1}{10}(F_0^{(1)} - F_1^{(1)})\Delta^2 + \frac{1}{120}(F_0^{(2)} + F_1^{(2)})\Delta^3 \tag{12}$$

$$r_1 = r_0 + \frac{1}{2}(v_0 + v_1)\Delta + \frac{9}{84}(F_0^{(0)} - F_1^{(0)})\Delta^2 + \frac{F_0^{(1)} + F_1^{(1)}}{84}\Delta^3 + \frac{F_0^{(2)} - F_1^{(2)}}{1680}\Delta^4 \tag{13}$$

Finally, $F_1^{(3)}$, $F_1^{(4)}$ are expressed as

$$F_1^{(3)} = \frac{60}{\Delta^3}(F_1^{(0)} - F_0^{(0)}) - \frac{12}{\Delta^2}(3F_1^{(1)} + 2F_0^{(1)}) + \frac{3}{\Delta}(3F_1^{(2)} - F_0^{(2)}) \tag{14}$$

$$F_1^{(4)} = \frac{360}{\Delta^4}(F_1^{(0)} - F_0^{(0)}) - \frac{1}{\Delta^3}(192F_1^{(1)} + 168F_0^{(1)}) + \frac{1}{\Delta^2}(36F_1^{(2)} - 24F_0^{(2)}) \tag{15}$$

2 8-th order

We solve these equations for $F_0^{(4)}, F_0^{(5)}, F_0^{(6)}, F_0^{(7)}$ as follows. By arrange the equations into matrix form, we have

$$\begin{pmatrix} \Delta^4/4! & \Delta^5/5! & \Delta^6/6! & \Delta^7/7! \\ \Delta^3/3! & \Delta^4/4! & \Delta^5/5! & \Delta^6/6! \\ \Delta^2/2! & \Delta^3/3! & \Delta^4/4! & \Delta^5/5! \\ \Delta & \Delta^2/2! & \Delta^3/3! & \Delta^4/4! \end{pmatrix} \begin{pmatrix} F_0^{(4)} \\ F_0^{(5)} \\ F_0^{(6)} \\ F_0^{(7)} \end{pmatrix} = \begin{pmatrix} F - (F_0 + F_0^{(1)}\Delta + F_0^{(2)}/2!\Delta^2 + F_0^{(3)}/3!\Delta^3) \\ F^{(1)} - (F_0^{(1)} + F_0^{(2)}\Delta + F_0^{(3)}/2!\Delta^2) \\ F^{(2)} - (F_0^{(2)} + F_0^{(3)}\Delta) \\ F^{(3)} - F_0^{(3)} \end{pmatrix} \tag{16}$$

$$F_0^{(4)} = \frac{-4(210(F_0^{(0)} - F_1^{(0)}) + 30(4F_0^{(1)} + 3F_1^{(1)})\Delta + 15(2F_0^{(2)} - F_1^{(2)})\Delta^2 + (4F_0^{(3)} + F_1^{(3)})\Delta^3)}{\Delta^4} \tag{17}$$

$$F_0^{(5)} = \frac{60(168(F_0^{(0)} - F_1^{(0)}) + (90F_0^{(1)} + 78F_1^{(1)})\Delta + (20F_0^{(2)} - 14F_1^{(2)})\Delta^2 + (2F_0^{(3)} + F_1^{(3)})\Delta^3)}{\Delta^5} \tag{18}$$

$$F_0^{(6)} = \frac{-120(420(F_0^{(0)} - F_1^{(0)}) + (216F_0^{(1)} + 204F_1^{(1)})\Delta + (45F_0^{(2)} - 39F_1^{(2)})\Delta^2 + (4F_0^{(3)} + 3F_1^{(3)})\Delta^3)}{\Delta^6} \tag{19}$$

$$F_0^{(7)} = \frac{840(120(F_0^{(0)} - F_1^{(0)}) + 60(F_0^{(1)} + F_1^{(1)})\Delta + 12(F_0^{(2)} - F_1^{(2)})\Delta^2 + (F_0^{(3)} + F_1^{(3)})\Delta^3)}{\Delta^7} \tag{20}$$

The explicit corrector are

$$r_1 = r_0 + v_0\Delta + \frac{13F_0^{(0)} + 5F_1^{(0)}}{36}\Delta^2 + \frac{17F_0^{(1)} - 10F_1^{(1)}}{252}\Delta^3 + \frac{7F_0^{(2)} + 5F_1^{(2)}}{1008}\Delta^4 + (\frac{F_0^{(3)}}{3024} - \frac{F_1^{(3)}}{3780})\Delta^5 \tag{21}$$

$$v_1 = v_0 + \frac{F_0^{(0)} + F_1^{(0)}}{2}\Delta + \frac{3(F_0^{(1)} - F_1^{(1)})}{28}\Delta^2 + \frac{F_0^{(2)} + F_1^{(2)}}{84}\Delta^3 + \frac{F_0^{(3)} - F_1^{(3)}}{1680}\Delta^4 \tag{22}$$

The implicit corrector for r is

$$r_1 = r_0 + \frac{v_0 + v_1}{2}\Delta + \frac{F_0^{(0)} - F_1^{(0)}}{9}\Delta^2 + \frac{F_0^{(1)} + F_1^{(1)}}{72}\Delta^3 + \frac{F_0^{(2)} - F_1^{(2)}}{1080}\Delta^4 + \frac{F_0^{(3)} + F_1^{(3)}}{30240}\Delta^5 \tag{23}$$

$$\tag{24}$$

Finally, $F_1^{(4)}$ and $F_1^{(5)}$ are expressed as

$$F_1^{(4)} = \frac{840}{\Delta^4}(F_0^{(0)} - F_1^{(0)}) + \frac{120}{\Delta^3}(3F_0^{(1)} + 4F_1^{(1)}) + \frac{60}{\Delta^2}(F_0^{(2)} - 2F_1^{(2)}) + \frac{4}{\Delta}(F_0^{(3)} + 4F_1^{(3)}) \quad (25)$$

$$F_1^{(5)} = \frac{10080}{\Delta^5}(F_0^{(0)} - F_1^{(0)}) + \frac{360}{\Delta^4}(13F_0^{(1)} + 15F_1^{(1)}) + \frac{120}{\Delta^3}(7F_0^{(2)} - 10F_1^{(2)}) + \frac{60}{\Delta^2}(F_0^{(3)} + 2F_1^{(3)}) \quad (26)$$

3 10-th order

Results are

$$v_1 = v_0 + \frac{F_0^{(0)} + F_1^{(0)}}{2}\Delta + \frac{F_0^{(1)} - F_1^{(1)}}{9}\Delta^2 + \frac{F_0^{(2)} + F_1^{(2)}}{72}\Delta^3 + \frac{F_0^{(3)} - F_1^{(3)}}{1008}\Delta^4 + \frac{F_0^{(4)} + F_1^{(4)}}{30240}\Delta^5 \quad (27)$$

$$r_1 = r_0 + \frac{v_0 + v_1}{2}\Delta + \frac{75600(F_0^{(0)} - F_1^{(0)})}{665280}\Delta^2 + \frac{10080(F_0^{(1)} + F_1^{(1)})}{665280}\Delta^3 \quad (28)$$

$$+\frac{840(F_0^{(2)} - F_1^{(2)})}{665280}\Delta^4 + \frac{42(F_0^{(3)} + F_1^{(3)})}{665280}\Delta^5 + \frac{F_0^{(4)} - F_1^{(4)}}{665280}\Delta^6 \quad (29)$$

$$F_1^{(5)} = \frac{-15120}{\Delta^5}(F_0^{(0)} - F_1^{(0)}) - \frac{(6720F_0^{(1)} + 8400F_1^{(1)})}{\Delta^4} - \frac{(1260F_0^{(2)} - 2100F_1^{(2)})}{\Delta^3} \quad (30)$$

$$-\frac{(120F_0^{(3)} + 300F_1^{(3)})}{\Delta^2} - \frac{(5F_0^{(4)} - 25F_1^{(4)})}{\Delta} \quad (31)$$

$$F_1^{(6)} = \frac{-302400}{\Delta^6}(F_0^{(0)} - F_1^{(0)}) - \frac{(141120F_0^{(1)} + 161280F_1^{(1)})}{\Delta^5} - \frac{(27720F_0^{(2)} - 37800F_1^{(2)})}{\Delta^4} \quad (32)$$

$$-\frac{(2760F_0^{(3)} + 4800F_1^{(3)})}{\Delta^3} - \frac{(120F_0^{(4)} - 300F_1^{(4)})}{\Delta^2} \quad (33)$$

$$F_1^{(7)} = \frac{-2721600}{\Delta^7}(F_0^{(0)} - F_1^{(0)}) - \frac{(1310400F_0^{(1)} + 1411200F_1^{(1)})}{\Delta^6} - \frac{(267120F_0^{(2)} - 317520F_1^{(2)})}{\Delta^5} \quad (34)$$

$$-\frac{(27720F_0^{(3)} + 37800F_1^{(3)})}{\Delta^4} - \frac{(1260F_0^{(4)} - 2100F_1^{(4)})}{\Delta^3} \quad (35)$$

Alternatatively, high-order force derivatives at $t = t_0 + \Delta/2$ are

$$F_{1/2}^{(5)} = \frac{-5670(F_0^{(0)} - F_1^{(0)})}{\Delta^5} - \frac{2835(F_0^{(1)} + F_1^{(1)})}{\Delta^4} \quad (36)$$

$$-\frac{1155(F_0^{(2)} - F_1^{(2)})}{2\Delta^3} - \frac{105(F_0^{(3)} + F_1^{(3)})}{2\Delta^2} - \frac{15(F_0^{(4)} - F_1^{(4)})}{8\Delta} \quad (37)$$

$$F_{1/2}^{(6)} = \frac{-2520(F_0^{(1)} - F_1^{(1)})}{\Delta^5} - \frac{1260(F_0^{(2)} + F_1^{(2)})}{\Delta^4} - \frac{240(F_0^{(3)} - F_1^{(3)})}{\Delta^3} - \frac{15(F_0^{(4)} + F_1^{(4)})}{\Delta^2} \quad (38)$$

$$F_{1/2}^{(7)} = \frac{453600(F_0^{(0)} - F_1^{(0)})}{\Delta^7} + \frac{226800(F_0^{(1)} + F_1^{(1)})}{\Delta^6} \quad (39)$$

$$+\frac{47880(F_0^{(2)} - F_1^{(2)})}{\Delta^5} + \frac{5040(F_0^{(3)} + F_1^{(3)})}{\Delta^4} + \frac{210(F_0^{(4)} - F_1^{(4)})}{\Delta^3} \quad (40)$$

4 Newmark Method

Following Kokubo & Makino (2004) (KM2004), we have derived the time-symmetric Newmark method (Newmark 1959) for the higher order Hermite integrators.

4.1 6-th order

With a coefficient α that we adjust to improve the integrator, the position and velocity at $t = t_1$ is expressed as

$$r_1 = r_0 + v_0\Delta + \sum_{i=2}^{6} \frac{F_0^{(i-1)}}{n!}\Delta^n + \alpha\frac{F_0^{(5)}}{7!}\Delta^7 \tag{41}$$

$$v_1 = v_0 + \sum_{i=1}^{6} \frac{F_0^{(i-1)}}{n!}\Delta^n \tag{42}$$

Put $F^{(3)}$, $F^{(4)}$, $F^{(5)}$ and we have the implicit form

$$v_1 = v_0 + \frac{1}{2}(F_0^{(0)} + F_1^{(0)})\Delta + \frac{1}{10}(F_0^{(1)} - F_1^{(1)})\Delta^2 + \frac{1}{120}(F_0^{(2)} + F_1^{(2)})\Delta^3 \tag{43}$$

$$r_1 = r_0 + A_1(v_0 + v_1)\Delta + A_2(F_0^{(0)} - F_1^{(0)})\Delta^2 + A_3(F_0^{(1)} + F_1^{(1)})\Delta^3 + A_4(F_0^{(2)} - F_1^{(2)})\Delta^4, \tag{44}$$

where

$$A_1 = \frac{1}{2}, A_2 = -\frac{60}{1680}(4\alpha - 7), A_3 = -\frac{20}{1680}(6\alpha - 7), A_4 = -\frac{1}{1680}(20\alpha - 21). \tag{45}$$

4.2 8-th order

The result is

$$v_1 = v_0 + \frac{F_0^{(0)} + F_1^{(0)}}{2}\Delta + \frac{3(F_0^{(1)} - F_1^{(1)})}{28}\Delta^2 + \frac{F_0^{(2)} + F_1^{(2)}}{84}\Delta^3 + \frac{F_0^{(3)} - F_1^{(3)}}{1680}\Delta^4 \tag{46}$$

$$r_1 = r_0 + A_1(v_0 + v_1)\Delta + A_2(F_0^{(0)} - F_1^{(0)})\Delta^2 + A_3(F_0^{(1)} + F_1^{(1)})\Delta^3 \tag{47}$$

$$+ A_4(F_0^{(2)} - F_1^{(2)})\Delta^4 + A_5(F_0^{(3)} + F_1^{(3)})\Delta^5, \tag{48}$$

where

$$A_1 = \frac{1}{2}, A_2 = \frac{1680}{30240}(5\alpha - 3), A_3 = \frac{420}{30240}(10\alpha - 9), A_4 = \frac{30}{30240}(28\alpha - 27), A_5 = \frac{1}{30240}(70\alpha - 69), \tag{49}$$

4.3 Derivation of α

The leading term of the local truncation errors of n-th order Hermite integrator with α term are given by

$$x_{\text{error}} = \frac{\alpha - 1}{(n+1)!}F_0^{(n-1)}\Delta^{n+1} \tag{50}$$

$$v_{\text{error}} = \frac{-1}{(n+2)!}F_0^{(n)}\Delta^{n+1}. \tag{51}$$

Here, we reduce the error in eccentricity (specifically eccentricity vector (e_x, e_y)). The equation (9) in Kokubo & Makino (2004) represents the leading error term in e_y

$$(e_y)_{\rm error} = \frac{\partial e_y}{\partial x} x_{\rm error} + \frac{\partial e_y}{\partial y} y_{\rm error} + \frac{\partial e_y}{\partial v_x} (v_x)_{\rm error} + \frac{\partial e_y}{\partial v_y} (v_y)_{\rm error} \tag{52}$$

We evaluate the integral of e_y for a half period and make it equal zero. The final result for n-th order integrator is

$$\alpha_n = 1 + \frac{1}{n+2} I_n \tag{53}$$

where I_n is a ratio between two integrals similar to equation (23) in KM2004. I_n is supposed to be 1 so that the optimal α is $(n+3)/(n+2)$.